4.14 SCALAR PRODUCT OF TWO VECTORS

26. *What are the different ways in which a vector can be multiplied by another vector ?*

These are *two* ways in which a vector can be multiplied by another vector :

(*i*) One way produces a scalar and is known as **scalar product.**

(*ii*) Another way produces a new vector and is known as **vector product.**

27. *Define scalar product of two vectors. Give its geometrical interpretation.*

Scalar or dot product. *The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and cosine of the angle θ between them.* Thus

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta = AB\cos\theta$$

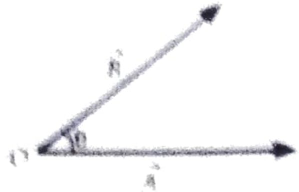

Fig. 4.44 Scalar product of $\vec{A}$ and $\vec{B}$ is a scalar :

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

As A, B and cos θ are all scalars, so the dot product of $\vec{A}$ and $\vec{B}$ is a scalar quantity. Both $\vec{A}$ and $\vec{B}$ have directions, but their dot product has no direction.

Geometrical interpretation of scalar product. As shown in Fig. 4.45(*a*), suppose two vectors $\vec{A}$ and $\vec{B}$ are represented by $\vec{OP}$ and $\vec{OQ}$ and $\angle POQ = \theta$.

$$\vec{A} \cdot \vec{B} = AB\cos\theta = A(B\cos\theta) = A(OR)$$

$= A \times$ Magnitude of component of $\vec{B}$ in the direction of $\vec{A}$

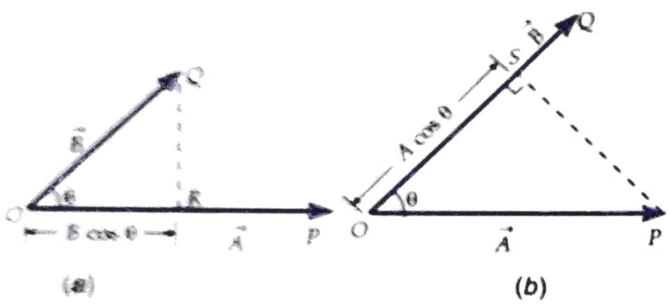

Fig. 4.45 (*a*) $B\cos\theta$ is the projection of $\vec{B}$ onto $\vec{A}$.
(*b*) $A\cos\theta$ is the projection of $\vec{A}$ onto $\vec{B}$.

From Fig. 4.45(*b*), we have

$$\vec{A} \cdot \vec{B} = AB\cos\theta = B(A\cos\theta)$$
$$= B(OS)$$

$= B \times$ Magnitude of component of $\vec{A}$ in the direction of $\vec{B}$.

Thus the scalar product of two vectors is equal to the product of magnitude of one vector and the magnitude of component of other vector in the direction of first vector.

28. *Give some examples of physical quantities that may be expressed as the scalar product of two vectors.*

Physical examples of scalar product of two vectors :

(*i*) **Work done (W).** *It is defined on the scalar product of the force ($\vec{F}$) acting on the body and the displacement ($\vec{s}$) produced.* Thus

$$W = \vec{F} \cdot \vec{s}$$

(*ii*) **Instantaneous power (P).** It is defined as the scalar product of force ($\vec{F}$) and the instantaneous velocity ($\vec{v}$) of the body. Thus

$$P = \vec{F} \cdot \vec{v}$$

(*iii*) **Magnetic flux (ϕ).** The magnetic flux linked with a surface is defined as the scalar product of magnetic induction ($\vec{B}$) and the area vector ($\vec{A}$). Thus

$$\phi = \vec{B} \cdot \vec{A}$$

NOTE As the scalar product of two vectors is a scalar quantity, so *work, power and magnetic flux are all scalar quantities.*

4.15 PROPERTIES OF SCALAR PRODUCT OF TWO VECTORS

29. *Mention important properties of the scalar product of vectors.*

Properties of scalar product :

(*i*) The scalar product is commutative *i.e.,*

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

(*ii*) The scalar product is distributive over addition *i.e.,*

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

(*iii*) If $\vec{A}$ and $\vec{B}$ are two vectors perpendicular to each other, then their scalar product is zero.

$$\vec{A} \cdot \vec{B} = A\,B\cos 90° = 0.$$

(*iv*) If $\vec{A}$ and $\vec{B}$ are two parallel vectors having same direction, then their scalar product has the maximum positive magnitude.

$$\vec{A} \cdot \vec{B} = A\,B\cos 0° = AB$$

(*v*) If $\vec{A}$ and $\vec{B}$ are two parallel vectors having opposite directions, then their scalar product has the maximum negative magnitude.

$$\vec{A} \cdot \vec{B} = A\,B\cos 180° = -A\,B$$

(*vi*) The scalar product of a vector with itself is equal to the square of its magnitude.

$$\vec{A} \cdot \vec{A} = A \cdot A\cos 0° = A \cdot A = A^2 = |\vec{A}|^2$$

(*vii*) Scalar product of two similar base vectors is unity and that of two different base vectors is zero.

$$\hat{i} \cdot \hat{i} = (1)(1)\cos 0° = 1$$

$$\therefore \quad \boxed{\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1}$$

$$\hat{i} \cdot \hat{j} = (1)(1) \cos 90^\circ = 0$$

$$\therefore \quad \boxed{\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0}$$

(*viii*) **Scalar product of two vectors $\vec{A}$ and $\vec{B}$ is equal to the sum of the products of their corresponding rectangular components.**

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

(*ix*) **The cosine of the angle θ between $\vec{A}$ and $\vec{B}$ is given by**

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$$

$$= \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

4.16 VECTOR PRODUCT OF TWO VECTORS

33. *Define vector or cross product of two vectors. How is its direction determined ?*

Vector or cross product. *The vector or cross product of two vectors is defined as the vector whose magnitude is equal to the product of the magnitudes of two vectors and sine of the angle between them and whose direction is perpendicular to the plane of the two vectors and is given by right hand rule.* Mathematically, if θ is the angle between vectors $\vec{A}$ and $\vec{B}$, then

$$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n}$$

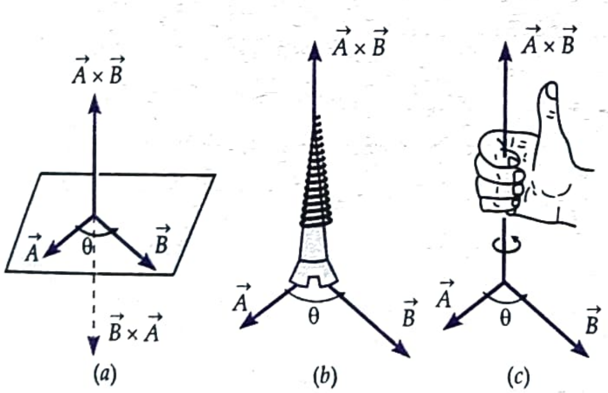

Fig. 4.49 Right hand rules for direction of vector product.

where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and its direction is given by right hand rule. Thus, the direction of $\vec{A} \times \vec{B}$ is same as that of unit vector $\hat{n}$.

Rules for determining the direction of $\vec{A} \times \vec{B}$:

(*i*) **Right handed screw rule.** As shown in Fig. 4.49(*b*), *if a right handed screw is placed with its axis perpendicular to the plane of vectors $\vec{A}$ and $\vec{B}$ and is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle, then the direction in which the screw advances gives the direction of $\vec{A} \times \vec{B}$.*

(*ii*) **Right hand thumb rule.** As shown in Fig. 4.49(*c*), *curl the fingers of the right hand in such a way that they point in the direction of rotation from vector $\vec{A}$ to $\vec{B}$ through the smaller angle, then the stretched thumb points in the direction of $\vec{A} \times \vec{B}$.*

34. *Give geometrical interpretation of vector product of two vectors.*

Geometrical interpretation of vector product. Suppose two vectors $\vec{A}$ and $\vec{B}$ are represented by the sides OP and OQ of a parallelogram $OPRQ$, as shown in Fig. 4.50.

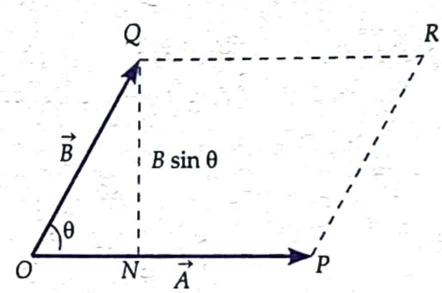

Fig. 4.50 Geometrical significance of vector product.

Let $\angle POQ = \theta$. Draw $QN \perp OP$. The magnitude of vector product $\vec{A} \times \vec{B}$ is

$$|\vec{A} \times \vec{B}| = AB \sin\theta = (OP)(OQ) \sin\theta$$
$$= (OP)(QN) \qquad [\because QN = OQ \sin\theta]$$
$$= \text{Area of parallelogram } OPRQ$$

Thus *the magnitude of the vector product of two vectors is equal to area of the parallelogram formed by the two vectors as its adjacent sides.* Moreover,

$$|\vec{A} \times \vec{B}| = 2 \times \tfrac{1}{2}(OP)(QN) = 2 \times \text{Area of } \Delta\, POQ$$

Thus *the magnitude of the vector product of two vectors is equal to twice the area of the triangle formed by the two vectors as its adjacent sides.*

35. *Give some examples of physical quantities which can be expressed as the vector product of two vectors.*

Physical examples of vector product :

(i) **Torque** $\vec{\tau}$. The torque acting on a particle is equal to the vector product of its position vector ($\vec{r}$) and force vector ($\vec{F}$). Thus

$$\vec{\tau} = \vec{r} \times \vec{F}$$

(ii) **Angular momentum** $\vec{L}$. The angular momentum of a particle is equal to the cross product of its position vector ($\vec{r}$) and linear momentum ($\vec{p}$). Thus

$$\vec{L} = \vec{r} \times \vec{p}$$

(iii) **Instantaneous velocity** $\vec{v}$. The instantaneous velocity of a particle is equal to the cross product of its angular velocity ($\vec{\omega}$) and the position vector ($\vec{r}$). Thus

$$\vec{v} = \vec{\omega} \times \vec{r}$$

4.17 PROPERTIES OF VECTOR PRODUCT

36. *Mention some important properties of vector product.*

Properties of vector product :

(i) **Vector product is anti-commutative** *i.e.*,

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

(ii) **Vector product is distributive over addition** *i.e.*,

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iii) **Vector product of two parallel or antiparallel vectors is a null vector.** Thus

$$\vec{A} \times \vec{B} = AB \sin(0° \text{ or } 180°)\,\hat{n} = \vec{0}$$

(iv) **Vector product of a vector with itself is a null vector.**

$$\vec{A} \times \vec{A} = AA \sin 0°\,\hat{n} = \vec{0}$$

(v) **The magnitude of the vector product of two mutually perpendicular vectors is equal to the product of their magnitudes.**

$$|\vec{A} \times \vec{B}| = AB \sin 90° = AB$$

(vi) **Vector product of orthogonal unit vectors.** The magnitude of each of the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ is 1 and the angle between any of two of them is 90°.

$$\therefore \quad \hat{i} \times \hat{j} = (1)(1) \sin 90°\,\hat{n} = \hat{n}$$

As $\hat{n}$ is a unit vector perpendicular to the plane of $\hat{i}$ and $\hat{j}$, so it is just the third vector $\hat{k}$

$$\therefore \quad \hat{i} \times \hat{j} = \hat{k}$$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \hat{i} \times \hat{k} = -\hat{j}$$

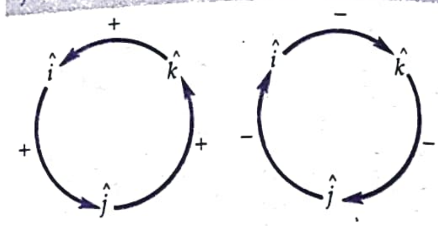

Fig. 4.51 Vector product of base vectors is cyclic (*a*) Anticlock-wise positive (*b*) Clockwise negative.

Aid to memory. Write $\hat{i}$, $\hat{j}$ and $\hat{k}$ cyclically round a circle, as shown in Fig. 4.51. Multiplying two unit vectors anticlockwise, we get positive value of third unit vector (*e.g.*, $\hat{i} \times \hat{j} = +\hat{k}$) ; and multiplying two unit vectors clockwise, we get negative value of third unit vector (*e.g.*, $\hat{j} \times \hat{i} = -\hat{k}$).

Also, $\hat{i} \times \hat{i} = (1)(1) \sin 0°\,\hat{n} = \vec{0}$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

(vii) **The vector product of two vectors can be expressed in terms of their rectangular components as a determinant.**

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

(viii) **Sine of the angle between two vectors.** If θ is the angle between two vectors $\vec{A}$ and $\vec{B}$, then

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$$

or

$$\sin\theta = \frac{|\vec{A} \times \vec{B}|}{|\vec{A}||\vec{B}|}$$

(ix) **Unit vector perpendicular to the plane of two vectors.** If $\hat{n}$ is a unit vector perpendicular to the plane of vectors $\vec{A}$ and $\vec{B}$, then

$$\vec{A} \times \vec{B} = AB \sin\theta\,\hat{n} = |\vec{A} \times \vec{B}|\,\hat{n}$$

or

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$