13.5 Bayes' Theorem

Consider that there are two bags I and II. Bag I contains 2 white and 3 red balls and Bag II contains 4 white and 5 red balls. One ball is drawn at random from one of the bags. We can find the probability of selecting any of the bags (i.e. $\frac{1}{2}$) or probability of drawing a ball of a particular colour (say white) from a particular bag (say Bag I). In other words, we can find the probability that the ball drawn is of a particular colour, if we are given the bag from which the ball is drawn. But, can we find the probability that the ball drawn is from a particular bag (say Bag II), if the colour of the ball drawn is given? Here, we have to find the reverse probability of Bag II to be selected when an event occurred after it is known. Famous mathematician, John Bayes' solved the problem of finding reverse probability by using conditional probability. The formula developed by him is known as *'Bayes theorem'* which was published posthumously in 1763. Before stating and proving the Bayes' theorem, let us first take up a definition and some preliminary results.

13.5.1 *Partition of a sample space*

A set of events $E_1, E_2, ..., E_n$ is said to represent a partition of the sample space S if

(a) $E_i \cap E_j = \phi, i \neq j, i, j = 1, 2, 3, ..., n$

(b) $E_1 \cup E_2 \cup ... \cup E_n = S$ and

(c) $P(E_i) > 0$ for all $i = 1, 2, ..., n$.

In other words, the events $E_1, E_2, ..., E_n$ represent a partition of the sample space S if they are pairwise disjoint, exhaustive and have nonzero probabilities.

As an example, we see that any nonempty event E and its complement E′ form a partition of the sample space S since they satisfy $E \cap E' = \phi$ and $E \cup E' = S$.

From the Venn diagram in Fig 13.3, one can easily observe that if E and F are any two events associated with a sample space S, then the set $\{E \cap F', E \cap F, E' \cap F, E' \cap F'\}$ is a partition of the sample space S. It may be mentioned that the partition of a sample space is not unique. There can be several partitions of the same sample space.

We shall now prove a theorem known as *Theorem of total probability*.

13.5.2 *Theorem of total probability*

Let $\{E_1, E_2, ..., E_n\}$ be a partition of the sample space S, and suppose that each of the events $E_1, E_2, ..., E_n$ has nonzero probability of occurrence. Let A be any event associated with S, then

$$P(A) = P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + ... + P(E_n)\, P(A|E_n)$$

$$= \sum_{j=1}^{n} P(E_j) P(A|E_j)$$

Proof Given that $E_1, E_2, ..., E_n$ is a partition of the sample space S (Fig 13.4). Therefore,

$$S = E_1 \cup E_2 \cup ... \cup E_n \quad ... (1)$$

and $\quad E_i \cap E_j = \phi,\ i \neq j,\ i, j = 1, 2, ..., n$

Now, we know that for any event A,

$$A = A \cap S$$
$$= A \cap (E_1 \cup E_2 \cup ... \cup E_n)$$
$$= (A \cap E_1) \cup (A \cap E_2) \cup ... \cup (A \cap E_n)$$

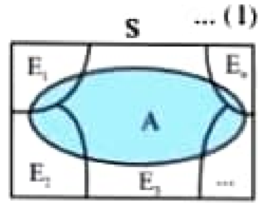

Fig 13.4

Also $A \cap E_i$ and $A \cap E_j$ are respectively the subsets of E_i and E_j. We know that E_i and E_j are disjoint, for $i \neq j$, therefore, $A \cap E_i$ and $A \cap E_j$ are also disjoint for all $i \neq j,\ i, j = 1, 2, ..., n$.

Thus, $\quad P(A) = P[(A \cap E_1) \cup (A \cap E_2) \cup ..... \cup (A \cap E_n)]$

$$= P(A \cap E_1) + P(A \cap E_2) + ... + P(A \cap E_n)$$

Now, by multiplication rule of probability, we have

$$P(A \cap E_i) = P(E_i)\, P(A|E_i) \text{ as } P(E_i) \neq 0 \forall i = 1, 2, ..., n$$

Therefore, $\quad P(A) = P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + ... + P(E_n) P(A|E_n)$

or $\quad P(A) = \sum_{j=1}^{n} P(E_j) P(A|E_j)$

Example 15 A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.

Solution Let A be the event that the construction job will be completed on time, and B be the event that there will be a strike. We have to find P(A).

We have

$$P(B) = 0.65,\ P(\text{no strike}) = P(B') = 1 - P(B) = 1 - 0.65 = 0.35$$
$$P(A|B) = 0.32,\ P(A|B') = 0.80$$

Since events B and B′ form a partition of the sample space S, therefore, by theorem on total probability, we have

$$P(A) = P(B)\, P(A|B) + P(B')\, P(A|B')$$
$$= 0.65 \times 0.32 + 0.35 \times 0.8$$
$$= 0.208 + 0.28 = 0.488$$

Thus, the probability that the construction job will be completed in time is 0.488.

We shall now state and prove the Bayes' theorem.